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# On absolute continuity of the spectrum of three- and four-dimensional periodic Schrödinger operators

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**Abstract**

We consider Schrödinger operators in $\mathbb{R}^n$, $n = 3, 4$, with electric potentials $V$ and magnetic potentials $A$ being periodic functions (with a common period lattice), and prove absolute continuity of the spectrum of the operators in question when $A \in H^q_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$, $2q > n - 1$, and when the function $|V|$ has relative bound zero with respect to the free Schrödinger operator $-\Delta$ in the sense of quadratic forms if $n = 3$ and the electric potential $V$ has relative bound zero with respect to the operator $-\Delta$ if $n = 4$.

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In this paper we deal with the problem of absolute continuity of the spectrum of the periodic Schrödinger operator

$$\widehat{H}(A, V) = \sum_{j=1}^{n} \left( -\mathrm{i}\frac{\partial}{\partial x_j} - A_j \right)^2 + V \tag{1}$$

acting on $L^2(\mathbb{R}^n)$, $n \geqslant 2$, where the electric potential $V : \mathbb{R}^n \to \mathbb{R}$ and the magnetic potential $A : \mathbb{R}^n \to \mathbb{R}^n$ are periodic functions with a common period lattice $\Lambda \subset \mathbb{R}^n$. The Schrödinger operator (1) (for $n = 3$ and $A \equiv 0$) plays an important role in the quantum solid state theory (see, e.g., [1, 2]). The spectrum of the operator $\widehat{H}(A, V)$ has a band-gap structure and absolute continuity of the spectrum implies the absence of eigenvalues (of infinite multiplicity) hence the spectral bands do not collapse into a point (see [2, 3]). The first result on absolute continuity of the spectrum of the Schrödinger operator (1) was obtained by Thomas in [4] for periodic electric potentials $V \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ (and $A \equiv 0$). In the last decade many papers were devoted to the problem of finding conditions on the electric potential $V$ and the magnetic potential $A$ which ensure absolute continuity of the spectrum. A survey on this subject is given in [5, 6]. The main results of the present paper are formulated in theorems 0.1 and 0.2. In particular, theorem 0.1 implies absolute continuity of the spectrum of operator (1) in the case $n = 3$ if the function $|V|$ has relative bound zero with respect to the free Schrödinger operator $\widehat{H}_0 \doteq -\Delta$ in the sense of quadratic forms and $A \in H^q_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$, $q > 1$.

Let $K$ be the fundamental domain of the lattice $\Lambda$, $\Lambda^*$ the reciprocal lattice with the basis vectors $E_j^*$ satisfying the conditions $(E_j^*, E_l) = \delta_{jl}$, where $\{E_l\}$ is the basis in the lattice $\Lambda$ and $\delta_{jl}$ is the Kronecker delta. We denote by $H^q(\mathbb{R}^n;\mathbb{C}^m)$, $m \in \mathbb{N}$, the Sobolev class of order $q \geqslant 0$. Let $\widetilde{H}^q(K;\mathbb{C}^m)$ be the set of functions $\phi : K \to \mathbb{C}^m$ whose $\Lambda$-periodic extensions belong to $H^q_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^m)$; $H^q(\mathbb{R}^n) = H^q(\mathbb{R}^n;\mathbb{C})$, $\widetilde{H}^q(\mathbb{R}^n) = \widetilde{H}^q(\mathbb{R}^n;\mathbb{C})$. In what follows, the functions defined on the fundamental domain $K$ will also be identified with their $\Lambda$-periodic extensions to all of $\mathbb{R}^n$.

A function $\mathcal{W} : \mathbb{R}^n \to \mathbb{C}$ is said to be *bounded with respect to the operator* $\widehat{H}_0 = -\Delta$ with the domain $D(\widehat{H}_0) = H^2(\mathbb{R}^n)$ if $\mathcal{W}\phi \in L^2(\mathbb{R}^n)$ for $\phi \in H^2(\mathbb{R}^n)$ and there exist numbers $\varepsilon \geqslant 0$ and $C_\varepsilon \geqslant 0$ such that

$$\|\mathcal{W}\phi\|^2 \leqslant \varepsilon^2 \|\widehat{H}_0\phi\|^2 + C_\varepsilon^2\|\phi\|^2 \tag{2}$$

for all $\phi \in H^2(\mathbb{R}^n)$. The infimum of numbers $\varepsilon$ in estimate (2) is called the *relative bound* of the function $\mathcal{W}$ with respect to the operator $\widehat{H}_0$ and will be denoted by $b_{\mathrm{op}}(\mathcal{W})$. If $\mathcal{W}|\phi|^2 \in L^1(\mathbb{R}^n)$ for $\phi \in H^1(\mathbb{R}^n)$ and there are numbers $\varepsilon \geqslant 0$ and $C_\varepsilon \geqslant 0$ such that

$$\left| \int_{\mathbb{R}^n} \mathcal{W}|\phi|^2 \,\mathrm{d}x \right| \leqslant \varepsilon \sum_{j=1}^n \left\| \frac{\partial \phi}{\partial x_j} \right\|^2 + C_\varepsilon \|\phi\|^2 \tag{3}$$

for all $\phi \in H^1(\mathbb{R}^n)$, then the function $\mathcal{W}$ is said to be $\widehat{H}_0$*-form bounded* (or bounded with respect to the operator $\widehat{H}_0$ in the sense of quadratic forms). The infimum of numbers $\varepsilon$ in estimate (3) is called the *relative* $\widehat{H}_0$*-form bound* of the function $\mathcal{W}$ and will be denoted by $b_{\mathrm{form}}(\mathcal{W})$. If a function $\mathcal{W}$ is bounded with respect to the operator $\widehat{H}_0$, then it is $\widehat{H}_0$-form bounded and $b_{\mathrm{form}}(\mathcal{W}) \leqslant b_{\mathrm{op}}(\mathcal{W})$ (moreover, in estimate (3), we can choose the same numbers $\varepsilon$ and $C_\varepsilon$ as in estimate (2)).

In the following, we shall consider potentials $V$ and $A$ such that $b_{\mathrm{form}}(V) < 1$ and $b_{\mathrm{form}}(|A|^2) = 0$. Under these conditions the quadratic form

$$W(A, V; \phi, \phi) = \sum_{j=1}^n \left\| \left( -\mathrm{i}\frac{\partial}{\partial x_j} - A_j \right)\phi \right\|^2 + \int_{\mathbb{R}^n} V|\phi|^2 \,\mathrm{d}x, \qquad \phi \in H^1(\mathbb{R}^n),$$

with the domain $Q(W(A, V; \cdot, \cdot)) = H^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is closed and bounded from below. By the KLMN theorem (see, e.g., [7]), the form $W(A, V; \cdot, \cdot)$ generates the self-adjoint operator (1) with some domain $D(\widehat{H}(A, V)) \subset H^1(\mathbb{R}^n)$.

The following two theorems are the main results of this paper.

**Theorem 0.1.** *Let $n = 3$ and let $V : \mathbb{R}^3 \to \mathbb{R}$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the periodic functions with a common period lattice $\Lambda \subset \mathbb{R}^3$. Suppose that the function $|V|$ is $\widehat{H}_0$-form bounded and $A \in \widetilde{H}^q(K;\mathbb{R}^3)$ for some $q > 1$. Then, there exists a number $C^{(3)}(A) \in (0, 1)$ such that under the condition $b_{\mathrm{form}}(|V|) \leqslant C^{(3)}(A)$ the spectrum of the periodic Schrödinger operator (1) is absolutely continuous.*

**Theorem 0.2.** *Let $n = 4$ and let $V : \mathbb{R}^4 \to \mathbb{R}$ and $A : \mathbb{R}^4 \to \mathbb{R}^4$ be the periodic functions with a common period lattice $\Lambda \subset \mathbb{R}^4$. Suppose that the electric potential $V$ is bounded with respect to the operator $\widehat{H}_0$ and $A \in \widetilde{H}^q(K;\mathbb{R}^4)$ for some $q > 3/2$. Then, there exists a number $C^{(4)}(A) \in (0, 1)$ such that under the condition $b_{\mathrm{op}}(V) \leqslant C^{(4)}(A)$ the spectrum of the periodic Schrödinger operator (1) is absolutely continuous.*

We let

$$\phi_N = v^{-1}(K) \int_K \phi(x)\,\mathrm{e}^{-2\pi\mathrm{i}(N,x)}\,\mathrm{d}x, \qquad N \in \Lambda^*,$$

denote the Fourier coefficients of functions $\phi \in L^1(K;\mathbb{C}^m)$, $m \in \mathbb{N}$, where $v(\cdot)$ is the Lebesgue measure on $\mathbb{R}^n$.

**Remark 1.** In theorems 0.1 and 0.2 we can choose more general classes of magnetic potentials $A$ which contain potentials $A \in \widetilde{H}^q(K;\mathbb{R}^n)$, $2q > n-1$. Let $n \geqslant 3$. For vectors $x \in \mathbb{R}^n\backslash\{0\}$ we shall use the notation

$$S_x^{n-2} \doteq \{\widetilde{e} \in S^{n-1} : (\widetilde{e}, x) = 0\},$$

where $S^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$. Let $\mathcal{B}(\mathbb{R})$ be the collection of Borel subsets $\mathcal{O} \subseteq \mathbb{R}$, $\mathfrak{M}$ the set of even signed Borel measures $\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ and $\mathfrak{M}_h$ the set of measures $\mu \in \mathfrak{M}$ such that

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}pt}\,\mathrm{d}\mu(t) = 1$$

for all $p \in (-h, h)$, $h > 0$. In particular, the sets $\mathfrak{M}_h$ contain the Dirac measure $\delta(\cdot)$. Fix a vector $\gamma \in \Lambda\backslash\{0\}$. Denote by $\mathcal{A}_\gamma(n,\Lambda)$ the class of magnetic potentials $A \in L^2(K;\mathbb{R}^n)$ that obey the following two conditions (see [8]):

($\mathbf{1}_\gamma$) the map

$$\mathbb{R}^n \ni x \to \{[0,1] \ni \xi \to A(x - \xi\gamma)\} \in L^2([0,1];\mathbb{R}^n)$$

is continuous;

($\mathbf{2}_\gamma$) there is a measure $\mu \in \mathfrak{M}_h$ (for some $h > 0$) such that

$$\max_{x\in\mathbb{R}^n}\max_{\widetilde{e}\in S_\gamma^{n-2}} \left| A_0 - \int_{\mathbb{R}} \mathrm{d}\mu(t) \int_0^1 A(x - \xi\gamma - t\widetilde{e})\,\mathrm{d}\xi \right| < \frac{\pi}{|\gamma|}, \tag{4}$$

where $A_0 = v^{-1}(K)\int_K A(x)\,\mathrm{d}x$ (and $|.|$ denotes the Euclidean norm on $\mathbb{R}^n$).

Theorems 0.1 and 0.2 are also valid if $A \in \mathcal{A}_\gamma(n,\Lambda)$ for some $\gamma \in \Lambda\backslash\{0\}$ (also see [8]). Condition ($\mathbf{1}_\gamma$) implies that $b_{\mathrm{form}}(|A|^2) = 0$. Condition ($\mathbf{2}_\gamma$) is fulfilled (under an appropriate choice of the vector $\gamma \in \Lambda\backslash\{0\}$ and the measure $\mu \in \mathfrak{M}_h, h > 0$) if $A \in \widetilde{H}^q(K;\mathbb{R}^n)$, $2q > n-2$ (see [9, 10]). If $2q > n-1$, then condition ($\mathbf{1}_\gamma$) is fulfilled as well. For the choice of Dirac measure $\mu = \delta$ in condition ($\mathbf{2}_\gamma$), inequality (4) is valid whenever

$$\sum_{N\in\Lambda^*\backslash\{0\}:(N,\gamma)=0} \|A_N\|_{\mathbb{C}^n} < \frac{\pi}{|\gamma|}. \tag{5}$$

Moreover, inequality (5) holds under an appropriate choice of the vector $\gamma \in \Lambda\backslash\{0\}$ if $\sum_{N\in\Lambda^*}\|A_N\|_{\mathbb{C}^n} < +\infty$ (see [9, 10]).

Denote by $L^p_w(K)$, $p \geqslant 1$, the (*weak*-$L^p(K)$) space of measurable functions $\mathcal{W} : K \to \mathbb{C}$ that satisfy the condition

$$\|\mathcal{W}\|_p^{(w)} \doteq \sup_{t>0} t(v(\{x \in K : |\mathcal{W}(x)| > t\}))^{1/p} < +\infty.$$

For $\mathcal{W} \in L^p_w(K)$, we also write

$$\|\mathcal{W}\|_{p,\infty}^{(w)} \doteq \overline{\lim_{t\to+\infty}}\, t(v(\{x \in K : |\mathcal{W}(x)| > t\}))^{1/p}.$$

In order to prove theorems 0.1 and 0.2, we apply the method suggested by Thomas in [4]. This method is a key point in the proof of absolute continuity of the spectrum of periodic elliptic differential operators. A survey of relevant results is contained in [5, 6] in which the generalized periodic Schrödinger operator

$$\sum_{j,l=1}^{n}\left(-\mathrm{i}\frac{\partial}{\partial x_j}-A_j\right)G_{jl}\left(-\mathrm{i}\frac{\partial}{\partial x_l}-A_l\right)+V,\qquad x\in\mathbb{R}^n, \tag{6}$$

is also considered (where the $\Lambda$-periodic matrix function $(G_{jl})_{j,l=1}^n$ with real entries is supposed to be symmetric and positive definite). The case of two-dimensional periodic Schrödinger operators has been studied in a comprehensive way. In particular, for $n = 2$, absolute continuity of the spectrum of the Schrödinger operator (1) was proved if the functions $V$ and $|A|^2$ are $\widehat{H}_0$-form bounded with relative $\widehat{H}_0$-form bounds zero (see [11] and also [12]). The generalized two-dimensional periodic Schrödinger operator (6) was considered in [11–16] (also see references therein). For $n \geqslant 3$, absolute continuity of the spectrum of the Schrödinger operator (1) was established by Sobolev (see [17]) for the periodic potentials $V \in L^p(K)$, $p > n-1$ and $A \in C^{2n+3}(\mathbb{R}^n;\mathbb{R}^n)$. These conditions on the potentials $V$ and $A$ were relaxed in subsequent papers [5, 6, 8, 10, 18, 19]. In [8], for $n \geqslant 3$, it was supposed that $V \in L_w^{n/2}(K)$, the magnetic potential $A$ satisfies conditions $(\mathbf{1}_\gamma)$ and $(\mathbf{2}_\gamma)$ from remark 1 (for some $\gamma \in \Lambda\backslash\{0\}$), and $\|V\|^{(w)}_{n/2,\infty} \leqslant C$, where $C = C(n; A) > 0$. The electric potential $V \in L_w^{n/2}(K)$ (for $A \equiv 0$) was also considered in [20]. The papers [21, 22] were addressed to the problem in question for the periodic electric potentials $V$ from the Kato class (for $n = 3$) and Morrey class respectively (also for $A \equiv 0$). If the periodic Schrödinger operator (1) has the period lattice $\Lambda = \mathbb{Z}^n$, $n \geqslant 3$, and is invariant under the transformation $x_1 \to -x_1$, then its spectrum is absolutely continuous under the conditions $V \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n)$ and $A \in L^q_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$, $q > n$ (see [23]). For $n \geqslant 3$, the generalized periodic Schrödinger operator (6) was also considered in [23–26].

In this paper we use some results obtained in [8]. Theorem 1.5 from section 1 is a particular case of theorem 3.1 from [8] which was proved as a consequence of statements concerning the periodic magnetic Dirac operator (see [27, 28]). Theorem 1.5 allows us to include the periodic magnetic potential $A$ into the Schrödinger operator (1).

The proof of theorems 0.1 and 0.2 is presented in section 1. Theorems 1.3 and 1.4 stated in section 1 are proved in section 2.

## 1. Proof of theorems 0.1 and 0.2

In the following, the scalar product and the norm on the space $L^2(K)$ will be written, omitting the notation $L^2(K)$ (unlike other spaces). Since $b_{\mathrm{form}}(|A|^2) = 0$ (see, e.g., [8]), one can define a sesquilinear form

$$W(A;k+\mathrm{i}\varkappa e;\psi,\phi)=\sum_{j=1}^{n}\left(\left(-\mathrm{i}\frac{\partial}{\partial x_j}-A_j+k_j-\mathrm{i}\varkappa e_j\right)\psi,\left(-\mathrm{i}\frac{\partial}{\partial x_j}-A_j+k_j+\mathrm{i}\varkappa e_j\right)\phi\right)$$

with the domain $Q(W(A;k+\mathrm{i}\varkappa e;\cdot,\cdot)) = \widetilde{H}^1(K) \subset L^2(K)$. In theorems 0.1 and 0.2, it is supposed that $b_{\mathrm{form}}(|V|) < 1$, therefore there exist numbers $\varepsilon \in (0,1)$ and $C_\varepsilon > 0$ such that the inequality

$$\int_{\mathbb{R}^n}|V|\cdot|\phi|^2\,\mathrm{d}x\leqslant\varepsilon\sum_{j=1}^{n}\left\|\frac{\partial\phi}{\partial x_j}\right\|^2_{L^2(\mathbb{R}^n)}+C_\varepsilon\|\phi\|^2_{L^2(\mathbb{R}^n)} \tag{7}$$

holds for all $\phi \in H^1(\mathbb{R}^n)$.

For $n = 4$, it is assumed that $b_{\text{op}}(V) < 1$. Hence, for some numbers $\varepsilon \in (0, 1)$ and $C_\varepsilon > 0$, the following inequality is valid for all $\phi \in H^2(\mathbb{R}^4)$:

$$\int_{\mathbb{R}^4} |V|^2|\phi|^2\,\mathrm{d}x \leqslant \varepsilon^2\|\widehat{H}_0\phi\|^2_{L^2(\mathbb{R}^4)} + C_\varepsilon^2\|\phi\|^2_{L^2(\mathbb{R}^4)}. \tag{8}$$

Inequality (8) and the interpolation of operators (see [7]) imply inequality (7) for $n = 4$ (with the same numbers $\varepsilon$ and $C_\varepsilon$).

From (7) it follows that the inequality

$$\int_K |V|\cdot|\phi|^2\,\mathrm{d}x \leqslant \varepsilon \sum_{j=1}^n \left\|\left(k_j - \mathrm{i}\frac{\partial}{\partial x_j}\right)\phi\right\|^2 + C_\varepsilon\|\phi\|^2 \tag{9}$$

is fulfilled for all $k \in \mathbb{R}^n$ and all $\phi \in \widetilde{H}^1(K)$. Therefore,

$$W(A, V; k + \mathrm{i}\varkappa e; \psi, \phi) \doteq W(A; k + \mathrm{i}\varkappa e; \psi, \phi) + \int_K V\overline{\psi}\phi\,\mathrm{d}x, \qquad \psi, \phi \in \widetilde{H}^1(K)$$

is a closed sectorial sesquilinear form generating an $m$-sectorial operator $\widehat{H}(A; k + \mathrm{i}\varkappa e) + V$ (with the domain $D(\widehat{H}(A; k + \mathrm{i}\varkappa e) + V) \subset \widetilde{H}^1(K) \subset L^2(K)$ independent of the complex vector $k + \mathrm{i}\varkappa e \in \mathbb{C}^n$). The operators $\widehat{H}(A; k) + V$ (for $\varkappa = 0$) are self-adjoint and have a compact resolvent. This implies that they have a discrete spectrum. For fixed vectors $k \in \mathbb{R}^n$ and $e \in S^{n-1}$, the operators $\widehat{H}(A; k + \zeta e) + V$, $\zeta \in \mathbb{C}$, form a self-adjoint analytic family of type (B) [29].

Let us denote

$$\widehat{H}_0(k + \mathrm{i}\varkappa e) = \sum_{j=1}^n \left(-\mathrm{i}\frac{\partial}{\partial x_j} + k_j + \mathrm{i}\varkappa e_j\right)^2,$$

$D(\widehat{H}_0(k + \mathrm{i}\varkappa e)) = \widetilde{H}^2(K) \subset L^2(K)$. For $n = 4$, from (8) it follows that $V\phi \in L^2(K)$ for all $\phi \in \widetilde{H}^2(K)$ and the estimate

$$\|V\phi\|^2 \leqslant \varepsilon^2\|\widehat{H}_0(k)\phi\|^2 + C_\varepsilon^2\|\phi\|^2 \tag{10}$$

holds for all $k \in \mathbb{R}^4$ and all $\phi \in \widetilde{H}^2(K)$.

The operator $\widehat{H}(A, V)$ is unitarily equivalent to the direct integral

$$\int^{\oplus}_{2\pi K^*} (\widehat{H}(A; k) + V)\frac{\mathrm{d}k}{(2\pi)^n v(K^*)},$$

where $K^*$ is the fundamental domain of the lattice $\Lambda^*$. The unitary equivalence is established via the Gel'fand transformation (see [2, 5]). Let $\lambda_j(k)$, $j \in \mathbb{N}$, be the eigenvalues of the operators $\widehat{H}(A; k) + V$, $k \in \mathbb{R}^n$. We assume that they are arranged in an increasing order (counting multiplicities). To prove absolute continuity of the spectrum of the operator $\widehat{H}(A, V)$, it suffices to find a vector $e \in S^{n-1}$ such that for all $k \in \mathbb{R}^n$ the functions $\mathbb{R} \ni \xi \to \lambda_j(k + \xi e)$, $j \in \mathbb{N}$, are not constant on every interval $(\xi_1, \xi_2) \subset \mathbb{R}$, $\xi_1 < \xi_2$ (see [2, 4]). If there exist a vector $k \in \mathbb{R}^n$, a number $\lambda \in \mathbb{R}$ and an index $j \in \mathbb{N}$ such that the equality $\lambda_j(k + \xi e) = \lambda$ is fulfilled for all $\xi \in (\xi_1, \xi_2)$, $\xi_1 < \xi_2$, then the analytic Fredholm theorem implies that the number $\lambda$ is an eigenvalue of the operators $\widehat{H}(A; k + \zeta e) + V$ for all $\zeta \in \mathbb{C}$. In theorems 1.1 and 1.2, for a given vector $\gamma \in \Lambda\backslash\{0\}$ it is proved that the operators $\widehat{H}(A; k + \mathrm{i}\varkappa|\gamma|^{-1}\gamma) + V - \lambda$ are invertible for all numbers $\lambda \in \mathbb{R}$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all sufficiently large numbers $\varkappa > 0$ (dependent on $\gamma$, $A$, $V$, and $\lambda \in \mathbb{R}$). Therefore, theorems 0.1 and 0.2 follow from theorems 1.1 and 1.2 respectively.

Fix a vector $\gamma \in \Lambda\backslash\{0\}$; $e = |\gamma|^{-1}\gamma \in S^{n-1}$. For vectors $x \in \mathbb{R}^n$, we write $x_\| \doteq (x, e)e$, $x_\perp \doteq x - (x, e)e$. For all $N \in \Lambda^*$, $k \in \mathbb{R}^n$ and $\varkappa \geqslant 0$, introduce the notation

$$G_N^\pm = G_N^\pm(k + \mathrm{i}\varkappa e) = (|k_\| + 2\pi N_\||^2 + (\varkappa \pm |k_\perp + 2\pi N_\perp|)^2)^{1/2}.$$

In what follows, we choose the vectors $k \in \mathbb{R}^n$ with $|(k,\gamma)| = \pi$. Hence, the following estimates are true: $G_N^- \geqslant \pi|\gamma|^{-1}$, $G_N^+ \geqslant \varkappa$, $G_N^+ \geqslant |k + 2\pi N|$ and $G_N^+ G_N^- \geqslant 2\pi|\gamma|^{-1}\varkappa$. The equality

$$\widehat{H}_0(k + \mathrm{i}\varkappa e)\phi = \sum_{N\in\Lambda^*} (k + 2\pi N + \mathrm{i}\varkappa e)^2 \phi_N \, \mathrm{e}^{2\pi \mathrm{i}(N,x)}, \qquad \phi \in \widetilde{H}^2(K),$$

holds, where $|(k + 2\pi N + \mathrm{i}\varkappa e)^2| = G_N^+ G_N^-$. Denote by $\widehat{L}^\theta = \widehat{L}^\theta(k + \mathrm{i}\varkappa e), \theta \in \mathbb{R}$, the non-negative operators acting on $L^2(K)$:

$$\widehat{L}^\theta \phi = \sum_{N\in\Lambda^*} (G_N^+ G_N^-)^\theta \phi_N \, \mathrm{e}^{2\pi \mathrm{i}(N,x)}, \qquad \phi \in D(\widehat{L}^\theta) = \begin{cases} \widetilde{H}^{2\theta}(K) & \text{if} \quad \theta > 0, \\ L^2(K) & \text{if} \quad \theta \leqslant 0. \end{cases}$$

For the operator $\widehat{L} = \widehat{L}^1$, one has $\|\widehat{L}\phi\| = \|\widehat{H}_0(k + \mathrm{i}\varkappa e)\phi\|$, $\phi \in D(\widehat{L}) = \widetilde{H}^2(K)$.

**Theorem 1.1.** *Let $n = 3$. Suppose that the periodic magnetic potential $A : \mathbb{R}^3 \to \mathbb{R}^3$ with a period lattice $\Lambda \subset \mathbb{R}^3$ belongs to the space $\widetilde{H}^q(K;\mathbb{R}^3)$ for some $q > 1$. Then there are numbers $C^{(3)}(A) \in (0,1)$, $C_1 = C_1(A) > 0$ and a vector $\gamma \in \Lambda\backslash\{0\}(e = |\gamma|^{-1}\gamma)$ such that for any $\Lambda$-periodic electric potential $V : \mathbb{R}^3 \to \mathbb{R}$ for which the function $|V|$ is $\widehat{H}_0$-form bounded and $b_{\mathrm{form}}(|V|) \leqslant C^{(3)}(A)$, and for any $\lambda \in \mathbb{R}$ there exists a number $\varkappa_0 > 0$ such that for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^3$ with $|(k,\gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$ the inequality*

$$\sup_{\psi\in\widetilde{H}^1(K):\|\widehat{L}^{1/2}(k+\mathrm{i}\varkappa e)\psi\|\leqslant 1} |W(A, V - \lambda; k + \mathrm{i}\varkappa e; \psi, \phi)| \geqslant C_1 \|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\| \tag{11}$$

*holds.*

**Theorem 1.2.** *Let $n = 4$. Suppose that the periodic magnetic potential $A : \mathbb{R}^4 \to \mathbb{R}^4$ with a period lattice $\Lambda \subset \mathbb{R}^4$ belongs to the space $\widetilde{H}^q(K;\mathbb{R}^4)$ for some $q > 3/2$. Then there exist numbers $C^{(4)}(A) \in (0,1)$, $C_1 = C_1(A) > 0$ and a vector $\gamma \in \Lambda\backslash\{0\}(e = |\gamma|^{-1}\gamma)$ such that for any $\Lambda$-periodic electric potential $V : \mathbb{R}^4 \to \mathbb{R}$ which is bounded with respect to the operator $\widehat{H}_0$ and satisfies the condition $b_{\mathrm{op}}(V) \leqslant C^{(4)}(A)$, and for any $\lambda \in \mathbb{R}$ there is a number $\varkappa_0 > 0$ such that for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^4$ with $|(k,\gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$ inequality (11) holds.*

Theorems 1.1 and 1.2 are proved at the end of this section. They are the consequences of theorems 1.3 and 1.4 respectively, and theorem 1.5.

**Theorem 1.3.** *Let $n = 3$ and let $\mathcal{W} : \mathbb{R}^3 \to \mathbb{R}$ be a periodic function with a period lattice $\Lambda \subset \mathbb{R}^3$. Suppose that the function $|\mathcal{W}|$ is $\widehat{H}_0$-form bounded, $\gamma \in \Lambda\backslash\{0\}$ (and $e = |\gamma|^{-1}\gamma$). Then for any $\delta > 0$, there is a number $\varkappa_0 > 0$ such that for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^3$ with $|(k,\gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$ the inequality*

$$\int_K |\mathcal{W}| \cdot |\phi|^2 \, \mathrm{d}x \leqslant C'(\delta + b_{\mathrm{form}}(|\mathcal{W}|)) \|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\|^2 \tag{12}$$

*is fulfilled, where $C' > 0$ is a universal constant.*

**Remark 2.** For $n = 3$, theorem 1.2 from [8] is a consequence of theorem 1.3. For $n \geqslant 3$, in theorem 1.2 from [8], it is proved that there exist numbers $\widetilde{C} = \widetilde{C}(n) > 0$ such that for any $\Lambda$-periodic function $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}$ which belongs to the space $L_w^n(K)$, and any vector $\gamma \in \Lambda\backslash\{0\}$ there is a number $\varkappa_0 > 0$ such that for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k,\gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$ the following inequality is valid:

$$\|\mathcal{W}\phi\| \leqslant \widetilde{C}\|\mathcal{W}\|_n^{(w)} \|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\|.$$

**Theorem 1.4.** *Let $n = 4$ and let $\mathcal{W} : \mathbb{R}^4 \to \mathbb{R}$ be a periodic function with a period lattice $\Lambda \subset \mathbb{R}^4$. Suppose that the function $\mathcal{W}$ is bounded with respect to the operator $\widehat{H}_0$, $\gamma \in \Lambda\backslash\{0\}$ (and $e = |\gamma|^{-1}\gamma$). Then for any $\delta > 0$, there is a number $\varkappa_0 > 0$ such that for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^4$ with $|(k, \gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^2(K)$ the inequality*

$$\|\mathcal{W}\phi\| \leqslant C''(\delta + b_{\mathrm{op}}(\mathcal{W}))\|\widehat{H}_0(k + \mathrm{i}\varkappa e)\phi\| \tag{13}$$

*holds, where $C'' > 0$ is a universal constant.*

**Remark 3.** Under the conditions of theorem 1.4 (for all $\varkappa \geqslant \varkappa_0$ and all vectors $k \in \mathbb{R}^4$ with $|(k, \gamma)| = \pi$) estimate (13) also implies the estimate

$$\int_K |\mathcal{W}| \cdot |\phi|^2 \,\mathrm{d}x \leqslant C''(\delta + b_{\mathrm{op}}(\mathcal{W}))\|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\|^2, \qquad \phi \in \widetilde{H}^1(K). \tag{14}$$

Indeed, from (13) it follows that

$$\|\mathcal{W}\widehat{L}^{-1}\psi\| \leqslant C''(\delta + b_{\mathrm{op}}(\mathcal{W}))\|\psi\|, \qquad \psi \in L^2(K).$$

The same estimate is true for the adjoint operator $(\mathcal{W}\widehat{L}^{-1})^*$. Hence, using the interpolation of operators (see, e.g., [7]), for all $\theta \in [0, 1]$, we derive

$$\|\widehat{L}^{-\theta}\mathcal{W}\widehat{L}^{\theta-1}\psi\| \leqslant C''(\delta + b_{\mathrm{op}}(\mathcal{W}))\|\psi\|, \qquad \psi \in \widetilde{H}^{2\theta}(K). \tag{15}$$

By continuity, the last inequality extends to all functions $\psi \in L^2(K)$. Choosing $\theta = 1/2$ in (15) and taking $\psi = \widehat{L}^{1/2}\phi$, we get estimate (14).

**Theorem 1.5.** *Let $n \geqslant 3$ and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a periodic magnetic potential with a period lattice $\Lambda \subset \mathbb{R}^n$. Suppose that $A \in \widetilde{H}^q(K; \mathbb{R}^n)$, $2q > n - 1$. Then, there exist a vector $\gamma \in \Lambda\backslash\{0\}$ ($e = |\gamma|^{-1}\gamma$) and numbers $C_2 = C_2(n, \Lambda, |\gamma|; A) > 0$ and $\varkappa_0 > 0$ such that for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$ the inequality*

$$\sup_{\psi \in \widetilde{H}^1(K):\|\widehat{L}^{1/2}(k+\mathrm{i}\varkappa e)\psi\| \leqslant 1} |W(A; k + \mathrm{i}\varkappa e; \psi, \phi)| \geqslant C_2\|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\| \tag{16}$$

*holds.*

Theorem 1.5 is proved in [8] for more general classes of periodic magnetic potentials. In theorem 1.3 from [8], for $n \geqslant 3$ and for $\Lambda$-periodic magnetic potentials $A \in \mathcal{A}_\gamma(n, \Lambda)$, $\gamma \in \Lambda\backslash\{0\}$, it is proved that there exist numbers $C_2 = C_2(n, \Lambda, |\gamma|; A) > 0$ and $\varkappa_0 > 0$ such that inequality (16) is fulfilled for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$. If $A \in \widetilde{H}^q(K; \mathbb{R}^n)$, $2q > n - 1$ ($n \geqslant 3$), then one can find a vector $\gamma \in \Lambda\backslash\{0\}$ such that $A \in \mathcal{A}_\gamma(n, \Lambda)$ (see [8–10]). Therefore, theorem 1.5 is a consequence of theorem 1.3 from [8].

**Proof of theorems 1.1 and 1.2.** Given the magnetic potential $A \in \widetilde{H}^q(K; \mathbb{R}^n)$, $2q > n - 1$, in accordance with theorem 1.5 we choose a vector $\gamma \in \Lambda\backslash\{0\}$ and a number $C_2$, such that estimate (16) holds for all sufficiently large numbers $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$. If $n = 3$, then $b_{\mathrm{form}}(|V - \lambda|) = b_{\mathrm{form}}(|V|)$ for all $\lambda \in \mathbb{R}$. Hence theorem 1.3 (estimate (12)) implies that for any $\delta > 0$ and for all sufficiently large numbers $\varkappa \geqslant \varkappa_0$, the following inequality is valid:

$$\int_K |V - \lambda| \cdot |\phi|^2 \,\mathrm{d}x \leqslant C'(\delta + b_{\mathrm{form}}(|V|))\|\widehat{L}^{1/2}\phi\|^2, \qquad \phi \in \widetilde{H}^1(K).$$

The polarization identity gives

$$\left|\int_K (V - \lambda)\overline{\psi}\phi \,\mathrm{d}x\right| \leqslant C'(\delta + b_{\mathrm{form}}(|V|))\|\widehat{L}^{1/2}\psi\| \cdot \|\widehat{L}^{1/2}\phi\|, \qquad \psi, \phi \in \widetilde{H}^1(K). \tag{17}$$

Let $C_1 \doteq \frac{1}{2}C_2$, $C^{(3)}(A) \doteq C_2(4C')^{-1}$. Then, taking any number $\delta \in (0, C^{(3)}(A)]$, from (16) and (17) we derive estimate (11). If $n = 4$, then $b_{\rm op}(V-\lambda) = b_{\rm op}(V)$ for all $\lambda \in \mathbb{R}$. Therefore, from theorem 1.5, remark 3 and the polarization identity, for any $\delta > 0$ and for all sufficiently large numbers $\varkappa \geqslant \varkappa_0$, we get

$$\left| \int_K (V-\lambda)\overline{\psi}\phi \, \mathrm{d}x \right| \leqslant C''(\delta + b_{\rm op}(V)) \|\widehat{L}^{1/2}\psi\| \cdot \|\widehat{L}^{1/2}\phi\|, \qquad \psi, \phi \in \widetilde{H}^1(K).$$

Let $C_1 \doteq \frac{1}{2}C_2$, $C^{(4)}(A) \doteq C_2(4C'')^{-1}$, and let us choose any number $\delta \in (0, C^{(4)}(A)]$. Now, theorem 1.2 immediately follows from the last estimate and estimate (16). □

## 2. Proof of theorems 1.3 and 1.4

First we introduce notation and prove some statements which will be used in the following.

Let $\Omega$ be a non-negative function from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \Omega(x)\,\mathrm{d}x = 1$. Denote

$$\Omega_\beta(x) \doteq \beta^{-n}\Omega\left(\frac{x}{\beta}\right), \qquad x \in \mathbb{R}^n, \quad \beta > 0;$$

$$C_{\beta,\Lambda}(\Omega) \doteq \max_{x\in\mathbb{R}^n} \sum_{N\in\Lambda} \Omega_\beta(x-N).$$

Given a function $\phi \in L^2(K)$ and a set $\mathcal{M} \subseteq \Lambda^*$, we define the function

$$\phi^{\mathcal{M}}(x) \doteq \sum_{N\in\mathcal{M}} \phi_N \, \mathrm{e}^{2\pi \mathrm{i}(N,x)}, \qquad x \in \mathbb{R}^n;$$

$\phi^{\Lambda^*} = \phi$, $\phi^{\emptyset} \equiv 0$ (the notation $\phi^{\mathcal{M}}$ will be used when either a set $\mathcal{M}$ or its complement $\Lambda^*\backslash\mathcal{M}$ is finite).

Fix a vector $\gamma \in \Lambda\backslash\{0\}$; $e = |\gamma|^{-1}\gamma$.

If $n = 3$, we set $\varepsilon = \delta + b_{\rm form}(|\mathcal{W}|)$. Then, for some number $C_\varepsilon > 0$, estimate (7) holds. Consequently estimate (9) holds for all $k \in \mathbb{R}^3$ as well.

If $n = 4$, we set $\varepsilon = \delta + b_{\rm op}(\mathcal{W})$. Then estimate (8) is also fulfilled for some number $C_\varepsilon > 0$. Consequently, for all $k \in \mathbb{R}^4$, estimate (10) is valid as well.

Let us pick a number $a > 0$ such that $C_\varepsilon \leqslant \varepsilon a^2$ (for both $n = 3$ and $n = 4$) and $a \geqslant 4\pi$ diam $K^*$, where diam $K^*$ is a diameter of the fundamental domain $K^*$.

We assume that $\varkappa_0 > 2a$ (and $\varkappa \geqslant \varkappa_0$). A few additional lower bounds on the number $\varkappa_0$ will be given below. For the vectors $k \in \mathbb{R}^n$, we suppose that $|(k,\gamma)| = \pi$.

Let

$$\mathcal{K} = \mathcal{K}(k;\varkappa) = \left\{ N \in \Lambda^* : G_N^- \leqslant \frac{\varkappa}{2} \right\}, \qquad \mathcal{K}_a = \mathcal{K}_a(k;\varkappa) = \{N \in \Lambda^* : G_N^- \leqslant a\} \subseteq \mathcal{K}.$$

If $N \in \mathcal{K}$, then $|k_\perp + 2\pi N_\perp| \geqslant \varkappa/2$. Let $\mathcal{N}(\mathcal{K}_a)$ be the number of vectors $N$ in the set $\mathcal{K}_a$. Since $a \geqslant 4\pi$ diam $K^*$, we have $a + 2\pi$ diam $K^* \leqslant 3a/2 < 3\varkappa/4$ and consequently

$$\mathcal{N}(\mathcal{K}_a) \leqslant c_0 v^{-1}(K^*) a^2 \varkappa^{n-2} = c_0 v(K) a^2 \varkappa^{n-2},$$

where $c_0 = c_0(n) > 0$.

For vectors $y \in \mathbb{R}^n$ with $y_\perp \neq 0$, we write

$$\widetilde{e}(y) \doteq \frac{y_\perp}{|y_\perp|} \in S_\gamma^{n-2}.$$

Denote

$$S_\gamma^{n-2}(\varkappa) \doteq \{y \in \mathbb{R}^n : (y,e) = 0 \text{ and } |y_\perp| = \varkappa\},$$

and choose vectors $y^{(j)} \in S_\gamma^{n-2}(\varkappa)$, $j = 1, \ldots, J$, such that the following two conditions are fulfilled:

(1) $|y^{(j_1)} - y^{(j_2)}| \geqslant a$ for different indices $j_1$ and $j_2$;
(2) for any vector $y \in S_\gamma^{n-2}(\varkappa)$, there exists a vector $y^{(j)}$ such that $|y - y^{(j)}| < a$.

For $a \leqslant \widetilde{a} \leqslant 2\varkappa$ and $y \in S_\gamma^{n-2}(\varkappa)$, denote by $\mathcal{N}(\widetilde{a}; y)$ the number of vectors $y^{(j)}$ satisfying the inequality $|y - y^{(j)}| \leqslant \widetilde{a}$.

We shall use (without proof) the following simple lemma.

**Lemma 2.1.** *There are constants* $c_1 = c_1(n) > 0$ *and* $c_2 = c_2(n) > c_1(n)$ *such that for all* $\widetilde{a} \in [a, 2\varkappa]$ *and all vectors* $y \in S_\gamma^{n-2}(\varkappa)$ *the estimates*

$$c_1\left(\frac{\widetilde{a}}{a}\right)^{n-2} \leqslant \mathcal{N}(\widetilde{a}; y) \leqslant c_2\left(\frac{\widetilde{a}}{a}\right)^{n-2}$$

*hold.*

In particular, from lemma 2.1 it follows that

$$J \leqslant c_2\left(\frac{2\varkappa}{a}\right)^{n-2}. \tag{18}$$

Define the sets

$$\mathcal{A}^{(j)} = \{N \in \mathcal{K}\backslash\mathcal{K}_a : |\varkappa\widetilde{e}(k + 2\pi N) - y^{(j)}| < G_N^-\}, \qquad j = 1, \ldots, J.$$

For any $N \in \mathcal{K}\backslash\mathcal{K}_a$, one has $G_N^- > a$. Therefore, by the choice of the vectors $y^{(j)} \in S_\gamma^{n-2}(\varkappa)$, we deduce that

$$\mathcal{K}\backslash\mathcal{K}_a = \bigcup_{j=1}^{J} \mathcal{A}^{(j)}.$$

If $N \in \mathcal{A}^{(j)}$, then

$$\begin{aligned} |(k + 2\pi N) - y^{(j)}|^2 &= |(k_\perp + 2\pi N_\perp) - y^{(j)}|^2 + |k_\| + 2\pi N_\||^2 \\ &\leqslant 2|\varkappa\widetilde{e}(k + 2\pi N) - y^{(j)}|^2 + 2|(k_\perp + 2\pi N_\perp) - \varkappa\widetilde{e}(k + 2\pi N)|^2 + |k_\| + 2\pi N_\||^2 \\ &< 2|\varkappa\widetilde{e}(k + 2\pi N) - y^{(j)}|^2 + 2(G_N^-)^2 < 4(G_N^-)^2. \end{aligned} \tag{19}$$

For vectors $N \in \mathcal{K}\backslash\mathcal{K}_a$ we shall use the short notation $\mathcal{N}_N \doteq \mathcal{N}(G_N^-; \varkappa\widetilde{e}(k + 2\pi N))$ for the number of indices $j \in \{1, \ldots, J\}$ such that $N \in \mathcal{A}^{(j)}$. The next estimate follows from lemma 2.1:

$$\mathcal{N}_N \geqslant c_1\left(\frac{G_N^-}{a}\right)^{n-2}. \tag{20}$$

Define the functions

$$\phi^{(j)} = \sum_{N \in \mathcal{A}^{(j)}} \mathcal{N}_N^{-1} \phi_N \,\mathrm{e}^{2\pi\mathrm{i}(N,x)}, \qquad j = 1, \ldots, J.$$

We have

$$\phi^{\mathcal{K}\backslash\mathcal{K}_a} = \sum_{j=1}^{J} \phi^{(j)}.$$

Now, let $n = 3$. By (19), for all $N \in \mathcal{A}^{(j)}$ we derive

$$\varepsilon|(k + 2\pi N) - y^{(j)}|^2 + C_\varepsilon < 4\varepsilon(G_N^-)^2 + \varepsilon a^2 < 5\varepsilon(G_N^-)^2.$$

Hence, from (9) (for all $j = 1, \dots, J$), it follows that

$$\int_K |\mathcal{W}| \cdot |\phi^{(j)}|^2 \, dx \leqslant v(K) \sum_{N \in \mathcal{A}^{(j)}} (\varepsilon |(k + 2\pi N) - y^{(j)}|^2 + C_\varepsilon) \big|\phi_N^{(j)}\big|^2$$
$$\leqslant 5\varepsilon v(K) \sum_{N \in \mathcal{A}^{(j)}} \mathcal{N}_N^{-2} (G_N^-)^2 |\phi_N|^2.$$

The last inequality and estimates (18) and (20) imply that

$$\int_K |\mathcal{W}| \cdot |\phi^{\mathcal{K}\setminus\mathcal{K}_a}|^2 \, dx \leqslant J \sum_{j=1}^{J} \int_K |\mathcal{W}| \cdot |\phi^{(j)}|^2 \, dx \leqslant 5\varepsilon J v(K) \sum_{j=1}^{J} \sum_{N \in \mathcal{A}^{(j)}} \mathcal{N}_N^{-2} (G_N^-)^2 |\phi_N|^2$$
$$= 5\varepsilon J v(K) \sum_{N \in \mathcal{K}\setminus\mathcal{K}_a} \mathcal{N}_N^{-1} (G_N^-)^2 |\phi_N|^2 \leqslant 10\varepsilon \frac{c_2}{c_1} \varkappa v(K) \sum_{N \in \mathcal{K}\setminus\mathcal{K}_a} G_N^- |\phi_N|^2$$
$$\leqslant 10\varepsilon \frac{c_2}{c_1} v(K) \sum_{N \in \mathcal{K}\setminus\mathcal{K}_a} G_N^+ G_N^- |\phi_N|^2 = 10\varepsilon \frac{c_2}{c_1} \|\widehat{L}^{1/2}(k + i\varkappa e)\phi^{\mathcal{K}\setminus\mathcal{K}_a}\|^2. \tag{21}$$

The function $|\mathcal{W}|$ is $\widehat{H}_0$-form bounded. In particular, this means that $\mathcal{W} \in L^1_{\text{loc}}(\mathbb{R}^3)$. For the functions

$$(\mathcal{W} * \Omega_\beta)(x) = \int_{\mathbb{R}^3} \mathcal{W}(y) \Omega_\beta(x - y) \, dy, \qquad x \in \mathbb{R}^3, \quad \beta > 0,$$

one has

$$\|\mathcal{W} * \Omega_\beta\|_{L^\infty(\mathbb{R}^3)} \leqslant C_{\beta,\Lambda}(\Omega) \|\mathcal{W}\|_{L^1(K)}, \tag{22}$$

and

$$\|\mathcal{W} - \mathcal{W} * \Omega_\beta\|_{L^1(K)} \to 0$$

as $\beta \to +0$. Let us pick a number $\beta > 0$ such that

$$\frac{|\gamma|}{\pi} c_0 a^2 \|\mathcal{W} - \mathcal{W} * \Omega_\beta\|_{L^1(K)} \leqslant \varepsilon. \tag{23}$$

We assume that the following constraint on the number $\varkappa_0$ is fulfilled:

$$\frac{|\gamma|}{\pi} C_{\beta,\Lambda}(\Omega) \|\mathcal{W}\|_{L^1(K)} \leqslant \varepsilon \varkappa_0. \tag{24}$$

Since

$$\|\phi^{\mathcal{K}_a}\|^2 \leqslant \frac{|\gamma|}{2\pi} \varkappa^{-1} \|\widehat{L}^{1/2}(k + i\varkappa e)\phi^{\mathcal{K}_a}\|^2,$$
$$\|\phi^{\mathcal{K}_a}\|^2_{L^\infty(\mathbb{R}^3)} \leqslant \mathcal{N}(\mathcal{K}_a) \sum_{N \in \mathcal{K}_a} |\phi_N|^2$$
$$\leqslant \frac{|\gamma|}{2\pi} c_0 a^2 v(K) \sum_{N \in \mathcal{K}_a} G_N^+ G_N^- |\phi_N|^2 = \frac{|\gamma|}{2\pi} c_0 a^2 \|\widehat{L}^{1/2}(k + i\varkappa e)\phi^{\mathcal{K}_a}\|^2,$$

using (22), (23) and (24), (for $\varkappa \geqslant \varkappa_0$) we obtain

$$\int_K |\mathcal{W}| \cdot |\phi^{\mathcal{K}_a}|^2 \, dx \leqslant \|\mathcal{W} - \mathcal{W} * \Omega_\beta\|_{L^1(K)} \|\phi^{\mathcal{K}_a}\|^2_{L^\infty(\mathbb{R}^3)} + \|\mathcal{W} * \Omega_\beta\|_{L^\infty(\mathbb{R}^3)} \|\phi^{\mathcal{K}_a}\|^2$$
$$\leqslant \varepsilon \|\widehat{L}^{1/2}(k + i\varkappa e)\phi^{\mathcal{K}_a}\|^2. \tag{25}$$

If $N \in \Lambda^* \setminus \mathcal{K}$, then $G_N^- > \frac{1}{3}|k + 2\pi N|$ and consequently

$$G_N^+ G_N^- > \tfrac{1}{3}|k + 2\pi N|^2. \tag{26}$$

Suppose that the next constraint on the number $\varkappa_0$ is also true:

$$\frac{|\gamma|}{\pi} C_\varepsilon \leqslant 2\varepsilon\varkappa_0. \tag{27}$$

Then (for $\varkappa \geqslant \varkappa_0$)

$$\begin{aligned}\int_K |\mathcal{W}| \cdot |\phi^{\Lambda^*\setminus\mathcal{K}}|^2 \, \mathrm{d}x &\leqslant v(K) \sum_{N\in\Lambda^*\setminus\mathcal{K}} (\varepsilon|k+2\pi N|^2 + C_\varepsilon)|\phi_N|^2 \\ &\leqslant 4\varepsilon v(K) \sum_{N\in\Lambda^*\setminus\mathcal{K}} G_N^+ G_N^- |\phi_N|^2 = 4\varepsilon \|\widehat{L}^{1/2}(k+\mathrm{i}\varkappa e)\phi^{\Lambda^*\setminus\mathcal{K}}\|^2. \end{aligned} \tag{28}$$

Since $\varepsilon = \delta + b_{\text{form}}(|\mathcal{W}|)$ and

$$\phi = \phi^{\mathcal{K}_a} + \phi^{\mathcal{K}\setminus\mathcal{K}_a} + \phi^{\Lambda^*\setminus\mathcal{K}}, \qquad \phi \in \widetilde{H}^1(K),$$

inequality (12) follows from (21), (25) and (28). This completes the proof of theorem 1.3.

Now, let $n = 4$. Using the inequality $C_\varepsilon \leqslant \varepsilon a^2$, from (19) for all $N \in \mathcal{A}^{(j)}$, $j = 1, \ldots, J$, we obtain

$$\varepsilon^2 |(k+2\pi N) - y^{(j)}|^4 + C_\varepsilon^2 < 16\varepsilon^2 (G_N^-)^4 + \varepsilon^2 a^4 < 17\varepsilon^2 (G_N^-)^4.$$

Hence, by (10),

$$\begin{aligned}\int_K |\mathcal{W}|^2 |\phi^{(j)}|^2 \, \mathrm{d}x &\leqslant v(K) \sum_{N\in\mathcal{A}^{(j)}} (\varepsilon^2 |(k+2\pi N) - y^{(j)}|^4 + C_\varepsilon^2) \big|\phi_N^{(j)}\big|^2 \\ &\leqslant 17\varepsilon^2 v(K) \sum_{N\in\mathcal{A}^{(j)}} \mathcal{N}_N^{-2} (G_N^-)^4 |\phi_N|^2. \end{aligned}$$

By analogy with (21) (also see (18) and (20)), we get

$$\begin{aligned}\int_K |\mathcal{W}|^2 |\phi^{\mathcal{K}\setminus\mathcal{K}_a}|^2 \, \mathrm{d}x &\leqslant J \sum_{j=1}^J \int_K |\mathcal{W}|^2 |\phi^{(j)}|^2 \, \mathrm{d}x \\ &\leqslant 17\varepsilon^2 J v(K) \sum_{j=1}^J \sum_{N\in\mathcal{A}^{(j)}} \mathcal{N}_N^{-2} (G_N^-)^4 |\phi_N|^2 = 17\varepsilon^2 J v(K) \sum_{N\in\mathcal{K}\setminus\mathcal{K}_a} \mathcal{N}_N^{-1} (G_N^-)^4 |\phi_N|^2 \\ &\leqslant 68\varepsilon^2 \frac{c_2}{c_1} \varkappa^2 v(K) \sum_{N\in\mathcal{K}\setminus\mathcal{K}_a} (G_N^-)^2 |\phi_N|^2 \leqslant 68\varepsilon^2 \frac{c_2}{c_1} v(K) \sum_{N\in\mathcal{K}\setminus\mathcal{K}_a} (G_N^+ G_N^-)^2 |\phi_N|^2 \\ &= 68\varepsilon^2 \frac{c_2}{c_1} \|\widehat{L}(k+\mathrm{i}\varkappa e)\phi^{\mathcal{K}\setminus\mathcal{K}_a}\|^2. \end{aligned} \tag{29}$$

Since the function $\mathcal{W}$ is bounded with respect to the operator $\widehat{H}_0$, we have $\mathcal{W} \in L^2_{\text{loc}}(\mathbb{R}^4) \subset L^1_{\text{loc}}(\mathbb{R}^4)$. As above (see (22)), for the functions $\mathcal{W} * \Omega_\beta$, $\beta > 0$, we derive the inequality

$$\|\mathcal{W} * \Omega_\beta\|_{L^\infty(\mathbb{R}^4)} \leqslant C_{\beta,\Lambda}(\Omega) \|\mathcal{W}\|_{L^1(K)}. \tag{30}$$

Moreover,

$$\|\mathcal{W} - \mathcal{W} * \Omega_\beta\|_{L^2(K)} \to 0$$

as $\beta \to +0$. Let us pick a number $\beta > 0$ such that

$$\frac{|\gamma|^2}{\pi^2} c_0 a^2 \|\mathcal{W} - \mathcal{W} * \Omega_\beta\|^2_{L^2(K)} \leqslant \varepsilon^2. \tag{31}$$

The number $\varkappa_0$ is supposed to satisfy inequality (24). The following estimates are valid:

$$\|\phi^{\mathcal{K}_a}\| \leqslant \frac{|\gamma|}{2\pi}\varkappa^{-1}\|\widehat{L}(k+\mathrm{i}\varkappa e)\phi^{\mathcal{K}_a}\|,$$

$$\|\phi^{\mathcal{K}_a}\|^2_{L^\infty(\mathbb{R}^4)} \leqslant \mathcal{N}(\mathcal{K}_a)\sum_{N\in\mathcal{K}_a}|\phi_N|^2$$

$$\leqslant \left(\frac{|\gamma|}{2\pi}\right)^2 c_0 a^2 v(K)\sum_{N\in\mathcal{K}_a}\left(G_N^+G_N^-\right)^2|\phi_N|^2 = \left(\frac{|\gamma|}{2\pi}\right)^2 c_0 a^2\|\widehat{L}(k+\mathrm{i}\varkappa e)\phi^{\mathcal{K}_a}\|^2.$$

Therefore, using (24), (30) and (31), for $\varkappa \geqslant \varkappa_0$, we obtain

$$\int_K |\mathcal{W}|^2|\phi^{\mathcal{K}_a}|^2\,\mathrm{d}x \leqslant 2\|\mathcal{W}-\mathcal{W}*\Omega_\beta\|^2_{L^2(K)}\|\phi^{\mathcal{K}_a}\|^2_{L^\infty(\mathbb{R}^4)} + 2\|\mathcal{W}*\Omega_\beta\|^2_{L^\infty(\mathbb{R}^4)}\|\phi^{\mathcal{K}_a}\|^2$$
$$\leqslant \varepsilon^2\|\widehat{L}(k+\mathrm{i}\varkappa e)\phi^{\mathcal{K}_a}\|^2. \tag{32}$$

Finally suppose that the number $\varkappa_0$ also satisfies inequality (27). Then estimates (10) and (26) yield

$$\int_K |\mathcal{W}|^2|\phi^{\Lambda^*\backslash\mathcal{K}}|^2\,\mathrm{d}x \leqslant v(K)\sum_{N\in\Lambda^*\backslash\mathcal{K}}\left(\varepsilon^2|k+2\pi N|^4 + C_\varepsilon^2\right)|\phi_N|^2$$
$$\leqslant 10\varepsilon^2 v(K)\sum_{N\in\Lambda^*\backslash\mathcal{K}}\left(G_N^+G_N^-\right)^2|\phi_N|^2 = 10\varepsilon^2\|\widehat{L}(k+\mathrm{i}\varkappa e)\phi^{\Lambda^*\backslash\mathcal{K}}\|^2. \tag{33}$$

Since $\varepsilon = \delta + b_{\mathrm{op}}(|\mathcal{W}|)$ and

$$\phi = \phi^{\mathcal{K}_a} + \phi^{\mathcal{K}\backslash\mathcal{K}_a} + \phi^{\Lambda^*\backslash\mathcal{K}}, \qquad \phi\in\widetilde{H}^2(K),$$

theorem 1.4 immediately follows from (29), (32) and (33).

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